MLC School

2016 TRIAL EXAMINATION

MATHEMATICS EXTENSION 2

Reading time 5 minutes

Writing time 3 hours

General Instructions:

- Write using black or blue pen
- Board approved calculators may be used
- A formulae booklet is provided

Total Marks 100

Section 1: 10 marks

- Attempt all questions.
- Write in blue or black pen.
- Answer on the multiple choice answer sheet provided.
- Allow approximately 15 minutes for this section.

Section 2: 90 marks

- Attempt all questions.
- Answer Section 2 on the writing booklets provided. Show all relevant mathematical reasoning and/or calculations.
- Allow about 2 hours and 45 minutes for this section.

Section 1

10 marks (Allow about 15 minutes)

Use the multiple Choice answer sheet for questions 1 – 10.

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1. Which of the following is an expression for $\int x^3 \ln x \, dx$?

A $\frac{1}{4}x^4 \ln x - \frac{1}{4}x^4 + c$

B $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$

C $\frac{1}{4}x^4 \ln x + \frac{1}{16}x^4 + c$

D $\frac{1}{4}x^4 \ln x + \frac{1}{4}x^4 + c$

2. Given $y = \sin^{-1}\left(e^x\right)$, an expression for $\frac{dy}{dx}$ is :

A $\operatorname{cosec} y$

B $\cot y$

C $\sec y$

D $\tan y$

3. Which of the following is the locus of the point P representing the complex number z moving on an Argand diagram, such that $|z - 2i| = 2 + \operatorname{Im} z$?

A a circle

B a hyperbola

C a parabola

D a straight line

4. The eccentricity of the ellipse $\frac{x^2}{k} + \frac{y^2}{k-1} = 1,\ k > 1$ is equal to

A $\frac{\sqrt{2k-1}}{k}$

B $\frac{1}{\sqrt{k}}$

C $\sqrt{\frac{2k-1}{k}}$

D $\frac{\sqrt{2k^2 - 2k + 1}}{k}$

5. The equations of the directrices of the hyperbola $\frac{x^2}{144} - \frac{y^2}{25} = 1$ are:

A $x = \pm \frac{13}{144}$

B $x = \pm \frac{13}{25}$

C $x = \pm \frac{25}{13}$

D $x = \pm \frac{144}{13}$

6. The gradient of the curve $x^2 y - xy^2 + 6 = 0$ at the point $P(2,3)$ is equal to:

A – 5

B $\frac{3}{8}$

C $\frac{9}{8}$

D 1

7. If $1+i$ is one root of the equation $z^2 - z + (1-i) = 0$, what is the value of the other root?

A $1-i$

B $-i$

C i

D $1+i$

8. The horizontal base of a solid is the circle $x^2 + y^2 = 1$. Each cross section taken perpendicular to the x – axis is a triangle with one side in the base of the solid. The length of this triangle side is equal to the altitude of the triangle through the opposite vertex. Which of the following is an expression for the volume of the solid?

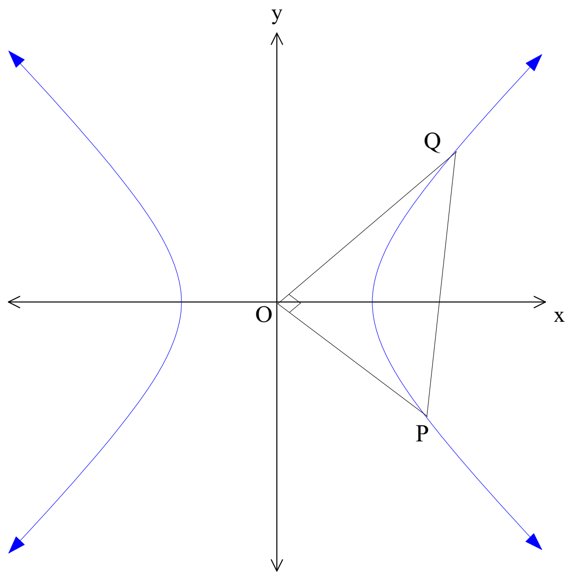
A $\frac{1}{2}\int_{-1}^{1}\left(1-x^2\right)dx$

B $\int_{-1}^{1}\left(1-x^2\right)dx$

C $\frac{3}{2}\int_{-1}^{1}\left(1-x^2\right)dx$

D $2\int_{-1}^{1}\left(1-x^2\right)dx$

9. The diagram below shows the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b>0$. The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\alpha, b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



Which of the following is correct?

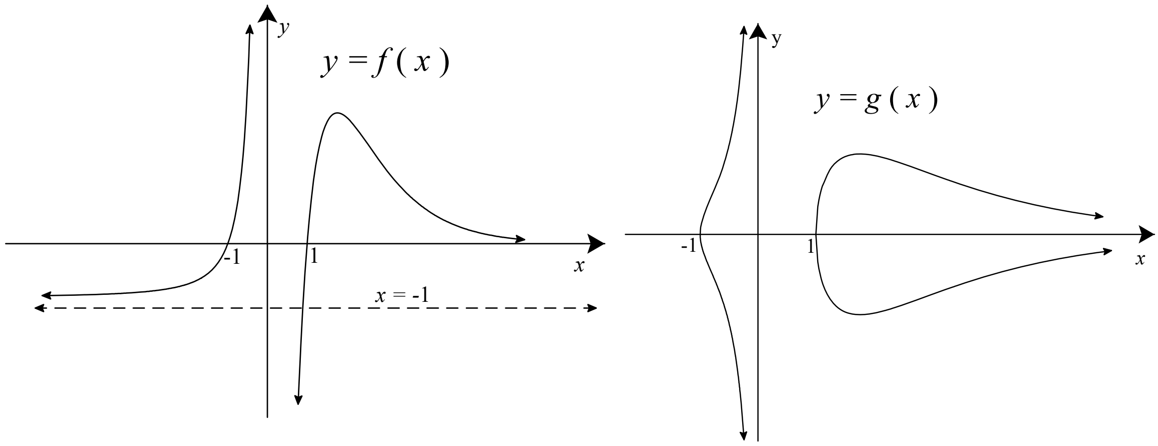
A. $\sin\theta\sin\alpha=-\frac{a^2}{b^2}$

B $\sin\theta\sin\alpha=\frac{a^2}{b^2}$

C $\tan\theta\tan\alpha=-\frac{a^2}{b^2}$

D $\tan\theta\tan\alpha=\frac{a^2}{b^2}$

10. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Which of the following best describes the relationship between $f(x)$ and $g(x)$?

A $|f(x)| = g(x)$

B $g(x) = \ln\left[f(x)\right]$

C $f(x) = \pm\ln|g(x)|$

D $g(x) = \pm\sqrt{f(x)}$

End of Section 1

Section 2

90 Marks (Allow about 2 hours and 45 minutes)

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available. Show all working and reasoning.

Question 11 (15 marks) **Use a SEPARATE writing booklet.**

(a) Suppose that *c* is a real number and that $z = c - i$. Express each of the following in the form $x + iy$ where *x* and *y* are real numbers.

(i) $\overline{(iz)}$

(ii) $\frac{1}{z}$ 2

(b) On an Argand diagram, shade the region specified by both $\text{Re}(z) \le 4$ and $|z - 4 + 5i| \le 3$. 2

(c) (i) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n \in Z^+$. 4

(ii) Given $w = \sqrt{3} - i$, express *w* in modulus argument form. 1

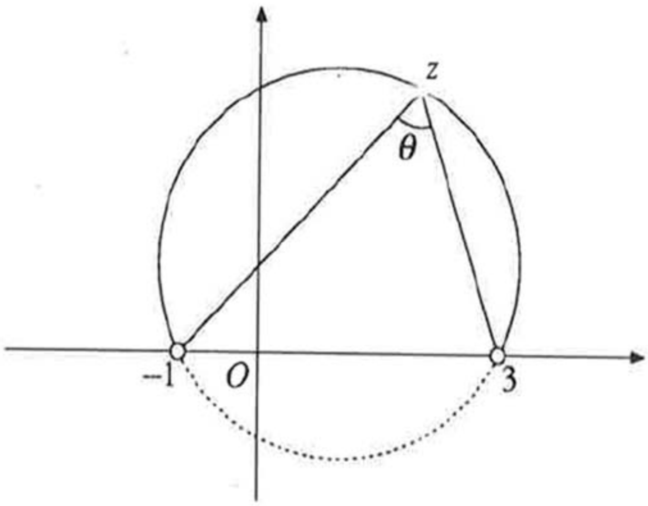
(iii) Hence express w^5 in the form $x + iy$ where *x* and *y* are real numbers. 2

Question 11 continues on page 7

Question 11 (continued)

(d) The diagram below shows the locus of point z in the complex plane such that

$$\arg(z-3)-\arg(z+1)=\frac{\pi}{3}$$



This locus is part of a circle. The angle between the chord from $(-1,0)$ to z and the chord from $(3,0)$ to z is θ, as shown.

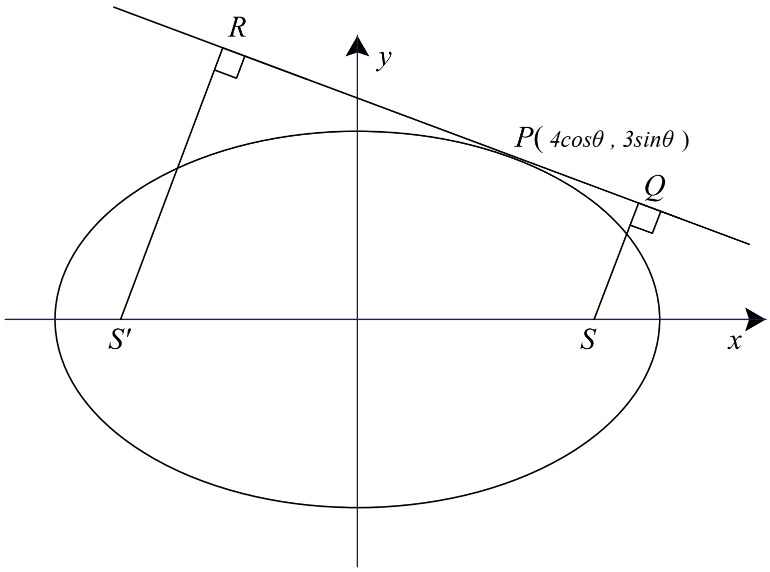
(i) Explain why $\theta=\frac{\pi}{3}$. 1

(ii) Find the centre of the circle. 3

End of Question 11

Question 12 (15 marks) **Use a SEPARATE writing booklet.**

(a) A tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is drawn at point $P(4\cos\theta, 3\sin\theta)$.
Perpendiculars are drawn from each focus of the ellipse to meet the tangent at Q and R as shown in the diagram.



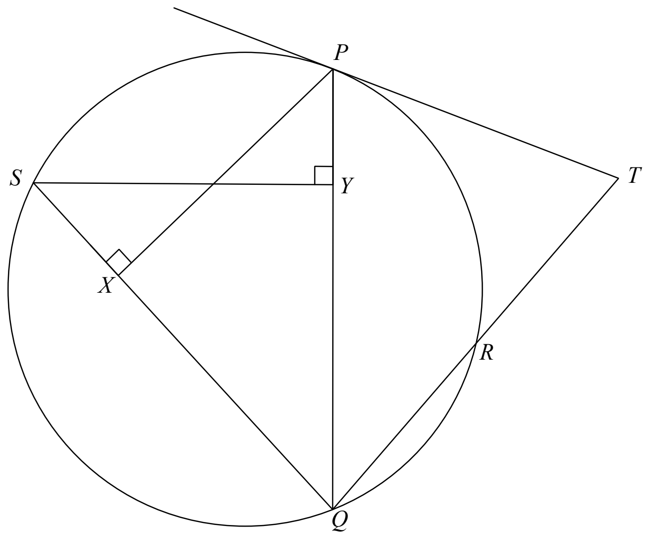
(i) Prove that the equation of the tangent at *P* is $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$ 2

(ii) Show that $QS \times RS' = 9$ 3

Question 12 continues on page 9

Question 12 (continued)

(b) In the diagram below, TP is the tangent of the circle at *P*, and *TQ* is a secant cutting the circle at *R*. *SQ* is a chord of the circle such that *PX* and *SY* are perpendicular to *SQ* and *PQ* respectively.



(i) Prove that $\angle TRP = \angle TPQ$ 3

(ii) Explain why *SPYX* is a cyclic quadrilateral and state the diameter of the circle. 1

(iii) Prove $\angle PYX = \angle PRQ$ 2

(c) (i) Show that the polynomial equation $4x^3 + 20x^2 - 23x + 6 = 0$ has a double root. 3

(ii) Hence find the value of each root. 1

End of Question 12

Question 13 (15 marks) **Use a SEPARATE writing booklet.**

(a) Consider the curve $x^3 + y^3 = 4xy$. Use implicit differentiation to find an expression for $\frac{dy}{dx}$. 2

(b) Find $\int \frac{5x\,dx}{x^2 + 3x + 6}$ 3

(c) Use the substitution $x = \frac{1}{u}$ to evaluate $\int_{\frac{1}{e}}^{e} \frac{\log_e x}{(1+x)^2}\,dx$ 3

(d) The equation $x^3 + px + 5 = 0$ has roots α, β and γ.

(i) Find in terms of p, $\alpha^2 + \beta^2 + \gamma^2$ 2

(ii) Show that $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = p^2$ 2

(iii) Find the cubic polynomial with integer coefficients, whose roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. 3

End of Question 13

Question 14 (15 marks) **Use a SEPARATE writing booklet.**

(a) (i) Show that $\sin x + \sin 5x = 2\sin 3x \cos 2x$ 2

(ii) Hence or otherwise, find all solutions of $\sin x + \sin 5x = 0$ for $o \le x \le 2\pi$. 3

(iii) Find $\int_0^{\pi} \sin 3x \cos 2x \, dx$ 2

(b) The hyperbola $\mathcal{H}: 16x^2 - 9y^2 = 144$ has foci $S(5,0)$ and $S(-5,0)$ and directrices $x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

(i) Find the equation of each asymptote of $\mathcal{H}$. 1

(ii) Show that the tangent to $\mathcal{H}$ at the point $P(3\sec\theta, 4\tan\theta)$, has equation $4x = (3\sin\theta)y + 12\cos\theta$. 3

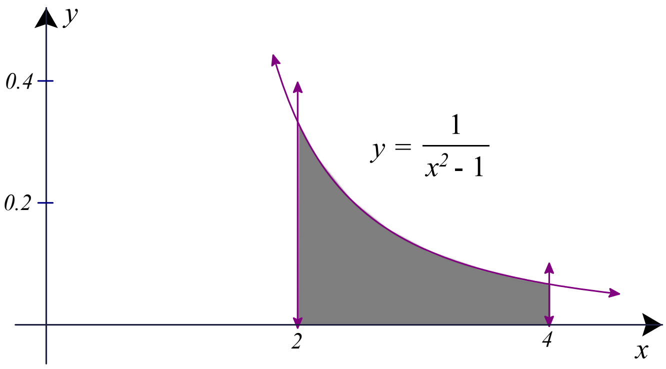
(iii) Q is the point of intersection of the tangent to $\mathcal{H}$ at P and the nearer directrix. Given that $0 < \theta < \frac{\pi}{2}$, show that Q has y – coordinate $\frac{12 - 20\cos\theta}{5\sin\theta}$. 1

(iv) Show that $\angle PSQ$ is a right angle. 3

End of Question 14

Question 15 (15 marks) **Use a SEPARATE writing booklet.**

(a) The region bounded by the curve $y = \dfrac{1}{x^2 - 1}$ and the x – axis between $x = 2$ and $x = 4$ is rotated through one revolution about the line $x = 2$.



(i) Use the method of cylindrical shells to show that the volume, V, of the solid formed is given by $V = 2\pi\displaystyle\int_2^4 \frac{x-2}{x^2-1}\,dx$. **2**

(ii) Hence find the exact value of V in simplest form. **3**

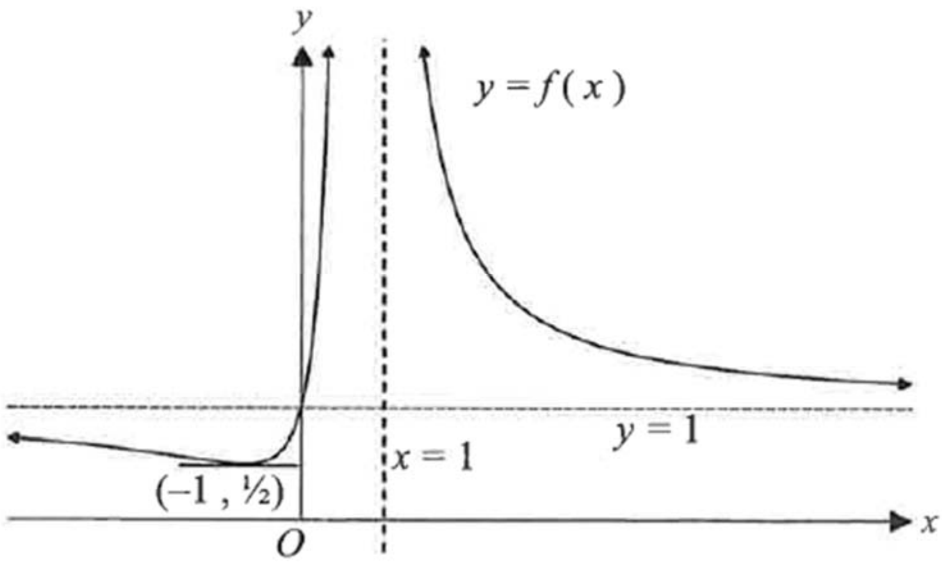
(b) The sequence of numbers A_n, where $n = 1,2,3....$, is defined by the following:

$A_1 = 0,\ A_2 = 1$ and $A_n = (n-1)\left(A_{n-1} + A_{n-2}\right)\ for\ n \geq 3$.

Use mathematical induction to show that $A_n = n!\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}\ for\ n \geq 1$ **4**

Question 15 continues on page 13

(c) The diagram below shows the graph of $y = f(x)$. The curve has a minimum turning point at $\left(-1, \frac{1}{2}\right)$.



On separate diagrams, sketch the following curves showing any important features.

(i) $y = f\left(\left|x-1\right|\right)$ 1

(ii) $y = \ln f(x)$ 1

(iii) $y = \dfrac{1}{f(x)}$ 2

(iv) $y = f'(x)$ 2

End of question 15

Question 16 (15 marks) **Use a SEPARATE writing booklet.**

(a) The points $T(2t,\frac{2}{t})$ and $R(2r,\frac{2}{r})$ lie on the rectangular hyperbola $xy = 4$.

(i) Find the midpoint M of TR. 1

(ii) It is known that TR is tangent to the parabola $x = \frac{p^2}{2}$, $y = p$.

Find the locus of M. 4

(b) (i) Using the property $\int_0^{2a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$ for $a > 0$, or otherwise,

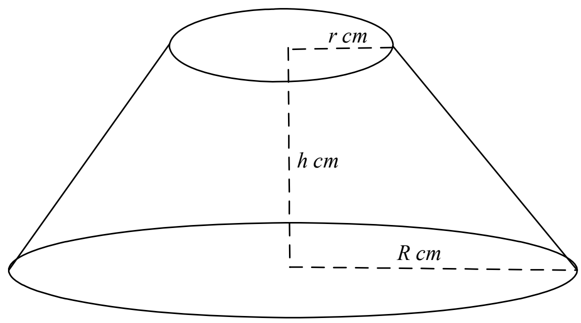
show that $\int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$ 2

(ii) Given $I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}}\,dx,\ n = 0,1,2,\ldots$, show that $I_n = \frac{n}{n+3} I_{n-2}$ for $n \geq 2$. 3

(iii) Hence evaluate $\int_0^1 \sqrt{x}(1-x)^{\frac{5}{2}}\,dx$. 1

(c) The frustrum shown below is obtained by cutting the top of a right circular cone. The top and bottom of the frustrum are circles with radii r cm and R cm respectively. The height of the frustrum is h cm. Use the method of slices to show that the volume of the frustrum is $\frac{\pi h}{3}\left(R^2 + Rr + r^2\right) cm^3$.

(Hint: turn the frustrum on its side) 4



End of Paper

Extension 2 2016 Trial.

① $\int x^3 \ln x \, dx$ B

Using integration by parts

$u = \ln x \quad v' = x^3$

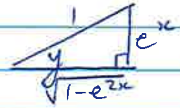
$u' = \frac{1}{x} \quad v = \frac{1}{4}x^4$

$\Rightarrow \frac{1}{4}\ln x \cdot x^4 - \frac{1}{4}\int x^3 dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

② $y = \sin^{-1} e^x$ D

$y' = \frac{e^x}{\sqrt{1-e^{2x}}}$



$y' = \frac{\sin y}{\cos y}$

$= \tan y$

③ $|z - 2i| = 2 + y$ C

$\sqrt{x^2 + (y-2)^2} = 2 + y$

$x^2 + y^2 - 4y + 4 = 4 + 4y + y^2$

$x^2 = 8y$

④ $a = \sqrt{k}, \quad b = \sqrt{k-1}$ B

$b^2 = a^2(1 - e^2)$

$1 - \frac{k-1}{k} = e^2$

$e = \sqrt{\frac{k-(k-1)}{k}}$

$= \sqrt{\frac{1}{k}}$

⑤ $\frac{b^2}{a^2} + 1 = e^2 \Rightarrow e^2 = \frac{25}{144} + 1$ D

$e = \frac{13}{12}$

$x = \frac{12}{13/12} = \frac{144}{13}$

⑥ $2xy + x^2 y' - y^2 - 2xy \cdot y' = 0$ B

$4 \times 3 + 4y' - 9 - 12y' = 0$

$-8y' = -3$

$y' = \frac{3}{8}$

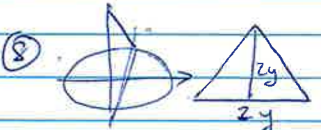
⑦ $\alpha + \beta = 1 + i + z = +1 \Rightarrow z = -i$ B

$\alpha\beta = z(1+i) = 1 - i$ (Check)

$(-i)(1+i) = -i + 1$

$= 1 - i$

⑧ D



$SA = \frac{1}{2} \cdot 2y \cdot 2y$

$SV = 2y^2 \delta x$

$= 2(1 - x^2)\delta x$

⑨ $m_{OP} = \frac{b \tan\theta}{a \sec\theta} = \frac{b \sin\theta}{a}$ A

$m_{OQ} = \frac{b \tan\alpha}{a \tan\alpha} = \frac{b \sin\alpha}{a}$

$m_{OP} \times m_{OQ} = -1$

$\Rightarrow \sin\alpha \sin\theta = -\frac{a^2}{b^2}$

⑩ D

Graph of $g(x)$ needs a ± since upper half reflected in bottom half and the log of $g(x)$ would be undefined as $x = \pm 1$ since $g(\pm 1) = 0$.

Section 2

⑪ (a) $z = c - i$

(i) $\overline{iz} = \overline{i(c-i)}$

$= \overline{ic + 1}$

$= -ic + 1$

$= 1 - ci$ 1

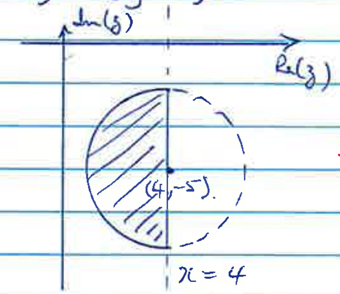
(ii) $\frac{1}{z} = \frac{1}{c-i} \times \frac{c+i}{c+i}$

$= \frac{c+i}{c^2+1}$

$= \frac{c}{c^2+1} + \frac{1}{c^2+1} i$ 1

(b) Need $x \le 4$ and

$(x-4)^2 + (y+5)^2 \le 9$.



2

(c)(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

When $n = 1$, LHS $= (\cos\theta + i\sin\theta)^1$

$= \cos\theta + i\sin\theta$

RHS $= \cos(1\theta) + i\sin(1\theta)$ 1

$= \cos\theta + i\sin\theta$

$=$ LHS $\Rightarrow$ true for $n = 1$

Assume true for $n = k$, $k \in Z^+$

ie $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ * 1

Prove true for $n = k+1$

$(\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ * 1

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$ ✓

(c)(i) Continued

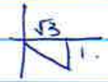
∴ True for $n = k+1$ if true for $n = k$.

But true for $n = 1$ ∴ by the principle of MI, true for all n. 1

ie $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin(n\theta)$.

(ii) $w = \sqrt{3} - i$

$= 2\text{cis}\left(-\frac{\pi}{6}\right)$ 1



(iii) $w^5 = 32\text{cis}\left(-\frac{5\pi}{6}\right)$

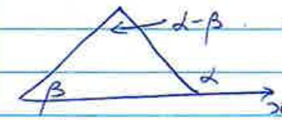
$= 32\left(\cos -\frac{5\pi}{6} + i\sin -\frac{5\pi}{6}\right)$ 1

$= 32\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -16(\sqrt{3} + i)$ 1

(d)(i) $\arg(z-3)$ is the angle the interval z to 3 makes with the x-axis (Say α)

$\arg(z+1)$ is the angle the interval z to -1 makes with the x-axis. (Say β) 1



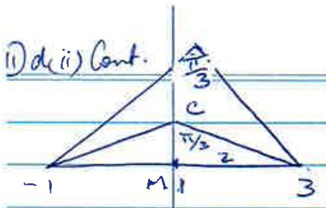
Since exterior angle of Δ = Sum of interior opposite $\theta = \alpha - \beta = \pi/3$.

(ii) Centre of chord is $(1, 0)$

∴ centre lies on line

$x = 1$ 1

1) d (ii) Cont.



$\tan \pi/3 = \frac{2}{CM}$

$CM = \frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$

$\therefore$ Centre $\left(1, \frac{2\sqrt{3}}{3}\right)$

(12) (a) (i). $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

$\Rightarrow \frac{x}{8} + \frac{2y}{9}y' = 0.$

$y' = \frac{-x}{8} \times \frac{9}{2y}$

$= \frac{-9x}{16y}$

$m = \frac{-36\cos\theta}{48\sin\theta}$

$= \frac{-3\cos\theta}{4\sin\theta}$

Eq^n of tangent

$y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$4\sin\theta\, y - 12\sin^2\theta = -3\cos\theta\, x + 12\cos^2\theta.$

$3x\cos\theta + 4y\sin\theta = 12$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$

(ii) $a = 4, b = 3, \quad e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}.$

S + S' $(\pm\sqrt{7}, 0)$.

$d_{RS'} = \left|\frac{-\sqrt{7}.3\cos\theta + 0 \times \sin\theta - 12}{\sqrt{9\cos^2\theta + 16\sin^2\theta}}\right|$

$= \left|\frac{-3\sqrt{7}\cos\theta - 12}{\sqrt{9\cos^2\theta + 16 - 16\cos^2\theta}}\right|$

$= \left|\frac{-3\sqrt{7}\cos\theta - 12}{\sqrt{16 - 7\cos^2\theta}}\right|.$

(3)

12(a)(ii) Cont.

$d_{QS} = \frac{|\sqrt{7}.3\cos\theta + 0 - 12|}{\sqrt{16 - 7\cos^2\theta}}$

$QS \times RS' = \left|\frac{-(3\sqrt{7}\cos\theta + 12)(3\sqrt{7}\cos\theta - 12)}{16 - 7\cos^2\theta}\right|$

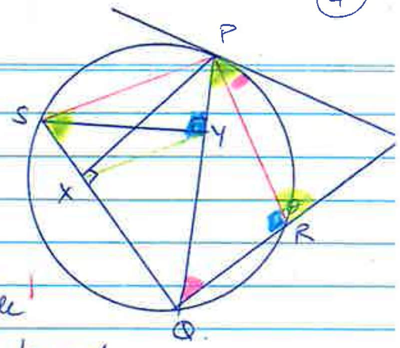
$= \frac{|-(63\cos^2\theta - 144)|}{16 - 7\cos^2\theta}$

$= \frac{144 - 63\cos^2\theta}{16 - 7\cos^2\theta}$

$= \frac{9(16 - 7\cos^2\theta)}{16 - 7\cos^2\theta}$

$= 9.$

(4)



(12) (b) (i) Join PS + PR

Let $\angle TRP = \theta$

$\angle PSQ = \angle TRP = \theta$ (exterior angle of cyclic quad = interior opposite)

$\angle PSQ = \angle TPQ = \theta$ angle in alternate segment equals angle between chord and tangent.

$\therefore \angle TPQ = \angle TRP = \theta$.

($\angle PTR = \angle PTQ$ (common angle)

$\therefore \Delta PTR \,|||\, \Delta PTQ$ equiangular)

(ii) SPYX is a cyclic quadrilateral with SP as diameter since $\angle SXP = \angle SYP = 90°$. angles in semicircle.

(iii). $\angle PYX + \angle PSX = 180°$ (opposite angles of cyclic quad SPYX)

$\therefore \angle PYX = (180 - \theta)°$

$\angle PRQ + \angle PRT = 180°$ (angles on a straight line)

$\therefore \angle PRQ = (180 - \theta)°$

$\therefore \angle PYX = \angle PRQ = (180 - \theta)°$.

(c). $P(x) = 4x^3 + 20x^2 - 23x + 6$

$P'(x) = 12x^2 + 40x - 23$

$= (6x + 23)(2x - 1)$

$P'\left(-\frac{23}{6}\right) = 0, \quad P'\left(\frac{1}{2}\right) = 0$

Test $P\left(-\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 20\left(\frac{1}{2}\right)^2 - 23\left(\frac{1}{2}\right) + 6$

$= \frac{1}{2} + 5 - 11\frac{1}{2} + 6 = 0$

$\therefore x = \frac{1}{2}$ is a double root

(ii) factor $2x - 1$

$\therefore$ Product of roots $= \frac{1}{2} \times \frac{1}{2} \times \alpha = -\frac{6}{4}$

$\alpha = -6$

Roots are $\frac{1}{2}, \frac{1}{2}, -6.$

13 (a) $x^3+y^3=4xy$

$\frac{d}{dx}\left(x^3+y^3=4xy\right)$

$3x^2+3y^2y' = 4y+4xy'$

$(3y^2-4x)y' = 4y-3x^2$

$\frac{dy}{dx} = \frac{4y-3x^2}{3y^2-4x}$.

(b) $\int \frac{5x\,dx}{x^2+3x+6} = \frac{5}{2}\int \frac{2x+3-3}{x^2+3x+6}\,dx$.

$= \frac{5}{2}\int \frac{2x+3}{x^2+3x+6}\,dx - \frac{15}{2}\int \frac{dx}{x^2+3x+6}$.

$= \frac{5}{2}\ln|x^2+3x+6| - \frac{15}{2}\int \frac{dx}{(x+\frac{3}{2})^2+\frac{15}{4}}$

$= \frac{5}{2}\ln|x^2+3x+6| - \frac{15}{2}\cdot\frac{1}{\sqrt{15}/2}\tan^{-1}\left(\frac{x+3/2}{\frac{\sqrt{15}}{2}}\right)+c$

$= \frac{5}{2}\ln|x^2+3x+6| - \sqrt{15}\tan^{-1}\left(\frac{2x+3}{\sqrt{15}}\right)+c$.

(c) $x = \frac{1}{u}$ $\quad dx = -\frac{1}{u^2}\,du$.

$x=e,\ u=\frac{1}{e}$

$x=\frac{1}{e}\quad u=e$.

$\therefore \int_{\frac{1}{e}}^{e} \frac{\ln x}{(1+x)^2}\,dx = \int_{e}^{\frac{1}{e}} \frac{\ln(\frac{1}{u})\cdot -\frac{1}{u^2}\,du}{(1+\frac{1}{u})^2}$.

$= -\int_{\frac{1}{e}}^{e} \frac{\ln u\,du}{(u+1)^2}$

$= -\int_{\frac{1}{e}}^{e} \frac{\ln x}{(x+1)^2}\,dx$

$\therefore 2\int_{\frac{1}{e}}^{e} \frac{\ln x}{(x+1)^2}\,dx = 0$.

$\Rightarrow \int_{\frac{1}{e}}^{e} \frac{\ln x}{(x+1)^2}\,dx = 0$.

13(d) $x^3+px+5=0$.

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0 - 2p$.

$= -2p$.

(ii) $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$

$= (p)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$.

$= p^2$

(iii) Roots of cubic $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$.

Sum of roots: $\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}$.

$= \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$.

$= -\frac{2p}{-5} \Rightarrow \frac{2p}{5}$.

Sum 2 at a time $= \frac{\alpha\beta}{\alpha\beta\gamma^2}+\frac{\alpha\gamma}{\alpha\beta^2\gamma}+\frac{\beta\gamma}{\alpha^2\beta\gamma}$

$= \frac{(\alpha\beta)^2+(\alpha\gamma)^2+(\beta\gamma)^2}{\alpha^2\beta^2\gamma^2}$

$= \frac{p^2}{(-5)^2}$

$= \frac{p^2}{25}$

Product of roots $= \frac{\alpha\beta\gamma}{\alpha^2\beta^2\gamma^2} = \frac{1}{\alpha\beta\gamma} = -\frac{1}{5}$.

$\therefore$ Eq^n. $x^3 - \frac{2p}{5}x^2 + \frac{p^2}{25}x + \frac{1}{5} = 0$.

$25x^3 - 10px^2 + p^2x + 5 = 0$.

(14) (a) (i) $\sin x + \sin 5x$

$= \sin(3x-2x) + \sin(3x+2x)$

$= \sin 3x\cos 2x - \cos 3x \sin 2x + \sin 3x \cos 2x + \cos 3x \sin 2x$

$= 2\sin 3x \cos 2x.$

(ii) $\sin x + \sin 5x = 2\sin 3x \cos 2x$

$\sin 3x \cos 2x = 0.$

$\sin 3x = 0.$ or $\cos 2x = 0$

$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$ $\quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}.$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ $\quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

(iii) $\int_0^{\pi} \sin 3x \cos 2x\, dx = \frac{1}{2}\int_0^{\pi} (\sin x + \sin 5x)\, dx$

$= \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_0^{\pi}$

$= \frac{1}{2}\left[1 + \frac{1}{5} - \left(-1 - \frac{1}{5}\right)\right]$

$= 1\frac{1}{5}.$

(b) $16x^2 - 9y^2 = 144.$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$ $\quad S(5,0)$ $S'(-5,0).$

$x = \frac{9}{5}, \; x = -\frac{9}{5}.$

$e^2 = \frac{b^2}{a^2} + 1 \Rightarrow \frac{16}{9} + 1.$ $\; e = \frac{5}{3}.$

(i) Asymptotes $y = \pm \frac{b}{a}x \Rightarrow y = \pm \frac{4}{3}x.$

(ii). At $P(3\sec\theta, 4\tan\theta)$ $\; \frac{dx}{d\theta} = 3\sec\theta\tan\theta.$ $\; \frac{dy}{d\theta} = 4\sec^2\theta$

$\frac{dy}{dx} = \frac{4\sec^2\theta}{3\sec\theta\tan\theta}$

$= \frac{4}{3\sin\theta}.$

Eq^n of tangent.

$y - 4\tan\theta = \frac{4}{3\sin\theta}(x - 3\sec\theta)$

$3y\sin\theta - 12\sin\theta\tan\theta = 4x - 12\sec\theta.$

$4x = 3y\sin\theta + 12(\sec\theta - \sin\theta\tan\theta)$

$= 3y\sin\theta + 12\left(\frac{1-\sin^2\theta}{\cos\theta}\right)$

$4x = 3y\sin\theta + 12\cos\theta.$

(14) (b) (iii). At Q. $x = \frac{9}{5}$

$4x = 3\sin\theta\, y + 12\cos\theta.$

$\frac{36}{5} = 3\sin\theta\, y + 12\cos\theta.$

$\frac{36 - 60\cos\theta}{15\sin\theta} = y.$

$\Rightarrow \frac{12 - 20\cos\theta}{5\sin\theta} = y.$

(iv). $P(3\sec\theta, 4\tan\theta)$ $\; S(5,0)$ $\; Q\left(\frac{9}{5}, \frac{12-20\cos\theta}{5\sin\theta}\right)$

$m_{PS} = \frac{4\tan\theta}{3\sec\theta - 5}$

$= \frac{4\sin\theta}{3 - 5\cos\theta}$

$m_{QS} = \frac{12 - 20\cos\theta}{5\sin\theta\left(\frac{9}{5} - 5\right)}$

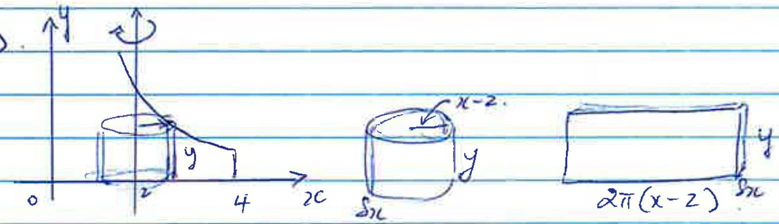
$= -\frac{1}{16}\frac{(12 - 20\cos\theta)}{\sin\theta}.$

$= -\left(\frac{3 - 5\cos\theta}{4\sin\theta}\right)$

$m_{PS} \times m_{QS} = -1$

$\therefore \angle PSQ = 90°.$

(15) (a)



Let δA be area of typical cylinder

$\delta A = 2\pi(x-2)y.$

$\delta V = 2\pi(x-2) \cdot \frac{1}{x^2-1}\,\delta x.$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=2}^{4} 2\pi\frac{(x-2)}{x^2-1}\,\delta x$

$= 2\pi\int_2^4 \frac{x-2}{x^2-1}\, dx$

(15) (a) (ii) Cont.

$$V = 2\pi\int_2^4 \left[\frac{x}{x^2-1} - \frac{2}{(x-1)(x+1)}\right] dx$$

$$= 2\pi\int_2^4 \frac{x}{x^2-1} - \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= 2\pi\left[\frac{1}{2}\ln|x^2-1| - \ln|x-1| + \ln|x+1|\right]_2^4$$

$$= \pi\left[\ln\left[\frac{(x+1)^3}{(x-1)}\right]\right]_2^4$$

$$= \pi\left(\ln\left(\frac{125}{3}\right) - \ln 27\right)$$

Volume $= \pi\ln\left(\frac{125}{81}\right)$ cu. units.

(b). $A_1 = 0 \quad A_2 = 1 \quad A_n = (n-1)(A_{n-1} + A_{n-2}) \quad n \geq 3$.

Need to prove $A_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

Given $A_1 = 0$, $\quad A_1 = 1!\sum_{k=0}^{1}\frac{(-1)^k}{k!} \qquad A_2 = 2!\sum_{k=0}^{2}\frac{(-1)^k}{k!}$

$A_2 = 1 \qquad = \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} \qquad = 2!\left[0 + \frac{(-1)^2}{2!}\right]$

$= \frac{0}{1} + \frac{-1}{1} = 0 \qquad = \frac{2!}{2!} \to 1.$

$\therefore$ A_n true for $n = 1, n = 2$.

Assume true for $n = r$, $r \geq 1$.

ie $A_r = r!\sum_{k=0}^{r}\frac{(-1)^k}{k!}$

Prove $A_{r+1} = (r+1)!\sum_{k=0}^{r+1}\frac{(-1)^k}{(k)!}$

$$\left(= r!\sum_{k=0}^{r}\frac{(-1)^k}{k!} + \frac{(-1)^n}{k!}(r+1)!\right)$$

Now $A_{r+1} = (r+1-1)(A_r + A_{r-1})$
$= r(A_r + A_{r-1})$

(15) (b). $A_{r+1} = r\left[r!\sum_{k=0}^{r}\frac{(-1)^k}{k!} + (r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\right]$

$$= r.(r-1)!\left[r\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + r.\frac{(-1)^r}{r!} + \sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\right]$$

$$= r(r-1)!\left[(r+1)\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + \frac{r(-1)^r}{r!}\right]$$

$$= (r+1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + r(-1)^r$$

$$= (r+1)!\left[\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + \frac{(r+1-1)(-1)^r}{(r+1)!}\right]$$

$$= (r+1)!\left[\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + \frac{(r+1)(-1)^r}{(r+1)!} - \frac{(-1)^r}{(r+1)!}\right]$$

$$= (r+1)!\left[\sum_{k=0}^{r-1}\frac{(-1)^k}{k!} + \frac{(-1)^r}{r!} + \frac{(-1)^{r+1}}{(r+1)!}\right]$$

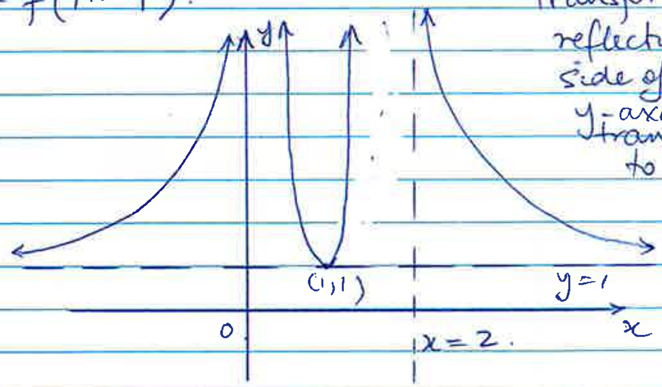
$$= (r+1)!\sum_{k=0}^{r+1}\frac{(-1)^k}{k!}$$

If A_r is true the A_{r+1} is also true for $r \geq 2$.

Since $A_1 + A_2$ are true, A_n is true for all $n \geq 1$.

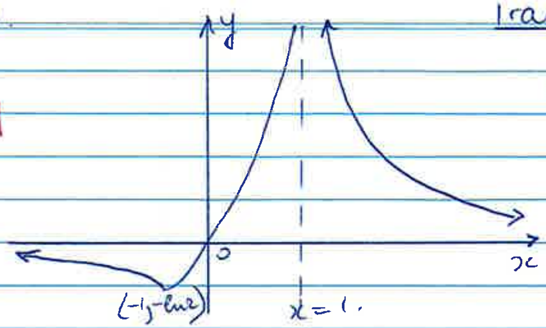
ie $A_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \quad n \geq 1$.

(15) (c) (i) $y = f(|x-1|)$.



Transformations:
reflection of RIGHT side of graph in y-axis
Translation 1 unit to the right.

15 (c) (ii)



Transformations: since

$f(x) > 0$ for all x,

$\ln f(x)$ exists for all x.

$\ln[f(0)] = \ln 1 = 0$.

$\ln[f(-1)] = \ln(\frac{1}{2}) = -\ln 2$

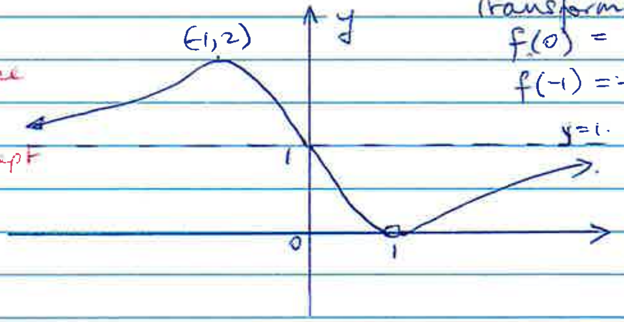
as $x \to \infty$, $\ln f(x) \to 0^+$

$x \to -\infty$, $\ln f(x) \to 0^-$

(iii)

1 for shape

1 for tpt. & intercept



Transformations:

$f(0) = 1 \therefore \frac{1}{f(0)} = 1 \Rightarrow (0,1)$.

$f(-1) = \frac{1}{2} \quad \frac{1}{f(-1)} = 2 \quad (-1, 2)$

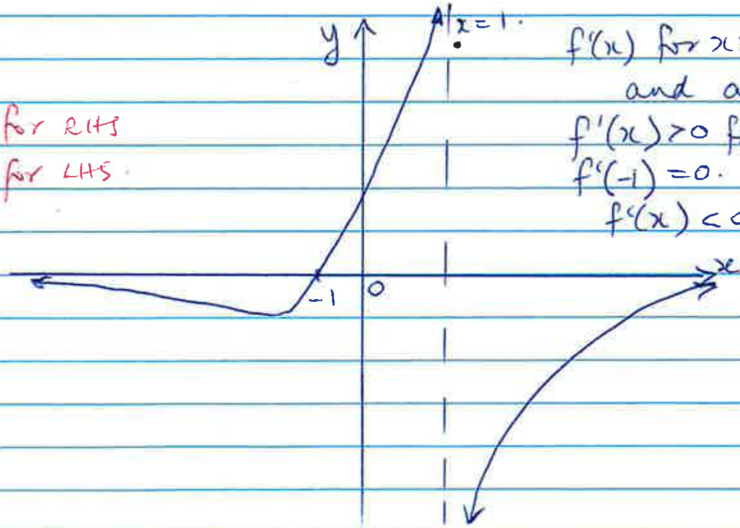
as $x \to 1$, $f(x) \to \infty$

$\therefore \frac{1}{f(x)} \to 0$.

(iv).

1 for RHS

1 for LHS.



$f'(x)$ for $x > 1$ is negative and approaches 0.

$f'(x) > 0$ for $-1 < x < 1$

$f'(-1) = 0$.

$f'(x) < 0$ for $x < -1$.

16 (a). $T(2t, \frac{2}{t})$ and $R(2r, \frac{2}{r})$ on $xy = 4$.

(i) $M. \left(t + r, \frac{1}{t} + \frac{1}{r}\right)$

(ii) $x = \frac{p^2}{2} \quad y = p$

$\frac{dx}{dp} = p \quad \frac{dy}{dp} = 1.$

$\frac{dy}{dx} = \frac{1}{p}.$

Equation of tangent to parabola

$y - p = \frac{1}{p}(x - \frac{p^2}{2})$

$py - p^2 = x - \frac{p^2}{2}$

$x - py = -\frac{p^2}{2}$. — (1)

Eq^n of TR. $m_{TR} = \frac{2(\frac{1}{t} - \frac{1}{r})}{2(t - r)} = -\frac{1}{tr}$.

$y - \frac{2}{t} = \frac{-1}{tr}(x - 2t)$.

$try - 2r = -x + 2t$

$x + try = 2(r + t)$. — (2).

Equating (1) + (2). $tr = -p$. — (i)

$2(r + t) = -\frac{p^2}{2}$. — (ii)

at M $x = t + r$, $y = \frac{t + r}{rt}$ (iii)

$= -\frac{p^2}{4}$.

$p = \sqrt{-4x}$

$\Rightarrow y = \frac{x}{rt}$.

$y = -\frac{x}{p}$

$y = \frac{-x}{\sqrt{-4x}}$.

$y^2 = \frac{x^2}{-4x}$

$y^2 = \frac{-x}{4}$ $\quad x \le 0$.

16 (b) (i) $\int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$

from the property $\int_0^1 \sqrt{x(1-x)}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{(\frac{1}{2}-x)(\frac{1}{2}+x)}\,dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{4}-x^2}\,dx$

This is the area of a semicircle radius $\frac{1}{2}$

$\therefore \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{4}-x^2}\,dx = \frac{1}{2}\times\pi\times(\frac{1}{2})^2$

$= \frac{\pi}{8}$

(ii) $I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}}\,dx$ $\qquad u=(1-x)^{\frac{n}{2}}$ $\quad v' = x^{\frac{1}{2}}$

$u' = -\frac{n}{2}(1-x)^{\frac{n-2}{2}}$ $\quad v = \frac{2}{3}x^{3/2}$

$= \left[\frac{2}{3}x^{3/2}(1-x)^{\frac{n}{2}}\right]_0^1 + \frac{n}{3}\int_0^1 x^{3/2}(1-x)^{\frac{n}{2}-1}\,dx$

$= 0 + \frac{n}{3}\int_0^1 x.\sqrt{x}(1-x)^{\frac{n}{2}-1}\,dx$

$= \frac{n}{3}\int_0^1 -(1-x-1)\sqrt{x}(1-x)^{\frac{n}{2}-1}\,dx$

$= -\frac{n}{3}\int_0^1 \left[\sqrt{x}(1-x)^{\frac{n}{2}} - \sqrt{x}(1-x)^{\frac{n-2}{2}}\right]dx$

$= -\frac{n}{3}.I_n + \frac{n}{3}I_{n-2}$

$(1+\frac{n}{3})I_n = \frac{n}{3}I_{n-2}$.

$I_n = \frac{n}{3}\times\frac{3}{3+n}I_{n-2}$

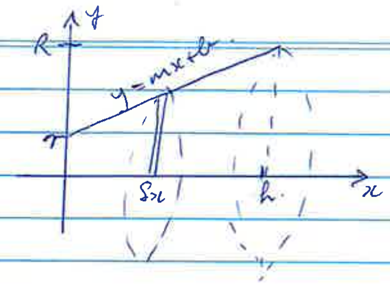
$= \frac{n}{n+3}I_{n-2}$.

(iv) $I_5 = \frac{5}{8}I_3 = \frac{5}{8}\times\frac{3}{6}I_1$

$I_1 = \int_0^1 \sqrt{x}(1-x)\,dx = \frac{\pi}{8}$ from (i).

$\therefore I_5 = \frac{5}{8}\times\frac{1}{2}\times\frac{\pi}{8} = \frac{5\pi}{128}$

16 (c)

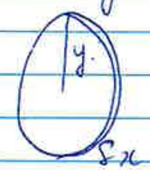


To find the equation of the line $y = mx+b$.

$m = \frac{R-r}{h-0} = \frac{R-r}{h}$ $\quad b = r$.

$\therefore y = \frac{R-r}{h}x + r$.

Taking a slice ⊥ to axis of rotation, Area of slice



$\delta A = \pi y^2$

$= \pi\left(\frac{R-r}{h}x+r\right)^2$

$\delta V = \pi\left(\frac{R-r}{h}x+r\right)^2\delta x$

$V = \lim_{\delta x\to 0}\sum_{x=0}^{h}\pi\left(\frac{R-r}{h}x+r\right)^2\delta x$

$= \pi\int_0^h \left(\frac{R-r}{h}x+r\right)^2 dx$

$= \frac{\pi}{3}\times\frac{h}{R-r}\left[\left(\frac{R-r}{h}x+r\right)^3\right]_0^h$

$= \frac{\pi h}{3(R-r)}\left[(R-r+r)^3 - r^3\right]$.

$= \frac{\pi h}{3(R-r)}(R^3-r^3)$

$= \frac{\pi h}{3(R-r)}(R-r)(R^2+Rr+r^2)$

Volume $= \frac{\pi h}{3}(R^2+Rr+r^2)$ cm^3.